ON STOKES PROBLEM FOR A THIRD-GRADE FLUID FOR GENERALIZED FRACTIONAL MODEL

*M. R. Mohyuddin**
*and Ehsan Ellahi Ashraf***

ABSTRACT

This paper studies the flows of a generalized third-grade fluid bounded by a plate(s). The fluid is electrically conducting, and Hall effects are taken into consideration. The governing non-linear partial differential equation is solved using Series and Fourier-transform methods. The effect of the variation in the material parameter and the magnetic field on the velocity field is discussed.

Keywords: Generalized third-grade fluid, unsteady flow, perturbation solution, oscillatory plate, Fourier transform.

INTRODUCTION

In an ionized gas, where the density is low and/or the magnetic field is very strong, the conductivity normal to the magnetic field is reduced due to the free spiraling of electrons and ions about the magnetic lines of force before suffering collisions; also, a current is induced in a direction normal to both the electric and magnetic fields. The phenomenon, well known in the literature, is called the Hall effect [11]. The study of magnetohydrodynamic flows with Hall currents has important engineering applications, particularly in the case of magnetohydrodynamic generators and of Hall accelerators, as well as in flight magnetohydrodynamics. The magnetohydrodynamic flow of a non-Newtonian fluid is of great importance in industrial and technological applications.

In recent years, fractional calculus has encountered much success in the description of complex dynamics. Fractional derivative models are used quite often to describe viscoelastic behavior of polymers in the glass-transition and the glassy state. The starting point is usually the classical differential equation, which is modified by replacing the classical, time-derivatives of an integer-order, by the so-called left-hand Liouville, or the Riemann-Liouville differential integral operators.

This generalization allows one to precisely define non-integer order integer or derivatives [1, 2]. Fractional derivative constitutive equations have been found to be quite flexible, in describing linear-viscoelastic behavior of polymers from glass-transition to the main or α relaxation in the glassy state. Recently, fractional calculus has encountered much success in the description of viscoelasticity [3, 4]. More recently, Tan and Xu [5] discussed the generalized second-grade flow, due to the impulsive motion of a flat plate. The second-grade fluid for a steady flow does not exhibit the property of shear thinning or thickening. For this reason, some experiments may be well described by the fluids of grade three or four [6, 7]. The model in the present paper is the generalized third-grade one.

The purpose of this work is to study the boundary-value problem, governing uni-directional unsteady flow, involving generalized third-grade fluid, with a view to emphasize the differences between the unsteady flow of a generalized third-grade fluid and the corresponding flow of a third-grade fluid, due to an oscillating flat plate The fractional calculus approach has been taken into account in the constitutive relationship of fluid model. By using the Fourier-transform of the sequential fractional derivatives, we obtain the solution for the flow.

FLOW EQUATIONS

The MHD equations governing the steady flow of an incompressible fluid are:

$$\rho \frac{d\mathbf{V}}{dt} = div\mathbf{T} + \mathbf{J} \times \mathbf{B}, \qquad 1$$

$$\nabla \cdot \mathbf{V} = 0, \qquad 2$$

$$\nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{B} = \mu_m \mathbf{J},\ \nabla \times \mathbf{E} = 0 \qquad 3$$

where ρ is the density, **J** is the current density, **B** is the total magnetic field, μ_m the magnetic permeability, **E** the total electric field-current and σ the electrical conductivity of the fluid. Making reference to Cowling

* Dept. of Mathematics, QAU, Islamabad. ** College of Aeronautical Engineering, National Uni. of Science and Technology, PAF Academy, Risalpur.

[9], when the strength of the magnetic field is very large, the generalized Ohm's law is modified to include the Hall-current, so that:

$$\mathbf{J} + \frac{\omega_e \tau_e}{\mathbf{B}_0}(\mathbf{J} \times \mathbf{B}) = \sigma\left[\mathbf{E} + \mathbf{V} \times \mathbf{B} + \frac{1}{en_e}\nabla p_e\right], \qquad 4$$

where ϖ_e is the cyclotron frequency of electrons, τ_e is the electron's collision-time, σ is the electrical conductivity, e is the electron-charge and p^e is the electron-pressure. The ion-slip and thermoelectric effects are not included in (4). Further, it is assumed that $\varpi_e \sim o(1)$ and $\varpi_i\tau_i$ <<1, where ϖ_i and τ_i are the cyclotron-frequency and collision-time for ions, respectively.

The constitutive equation for the Cauchy stress-tensor of third-grade fluid is [12]:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(tr\mathbf{A}_2)\mathbf{A}_1. \qquad 5$$

In Eq. 5, μ is the dynamic viscosity, α_1,α_2, β_1,β_2, and β_3 are the material constant and the kinematic tensors $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are given as follows:

$$\mathbf{A}_1 = (grad\mathbf{V}) + (grad\mathbf{V})^{\top}, \qquad 6$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(grad\mathbf{V}) + (grad\mathbf{V})^{\top}\mathbf{A}_{n-1},\ n = 2,3, \qquad 7$$

where **V** is the velocity-field, *grad* is the gradient-operator, the transpose and *d*/*dt* is the material time derivative. A detailed thermodynamic analysis of the model, represented by Eq. (5) is given by Fosdick and Rajagopal [6]. They showed that if all the motions of the fluid are to be compatible with thermodynamics, in the sense that these motions meet the Clausius-Duhem inequality, and if it is assumed that the specific Helmholtz free energy is a minimum when the fluid is locally at rest, then:

$$\mu \geq 0,\ \ \alpha_1 \geq 0, \qquad 8$$
$$|\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},$$
$$\beta_1 = \beta_2 = 0,\ \beta_3 \geq 0$$

and Eq. 5 for a thermodynamically compatible fluid of third-grade becomes

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1. \qquad 9$$

For generalized third-grade fluids, the constitutive relation is Eq. 9, but **A***n* is defined as follows [3, 4, 5, 8]:

$$\mathbf{A}_n = D_t^{\beta}\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(grad\mathbf{V}) + (grad\mathbf{V})^{\top}\mathbf{A}_{n-1},\ n = 2,3, \qquad 10$$

where D^{β}_{τ} is Riemann-Liouville fractional calculus operator and is defined by [8].

$$D_t^{\beta}f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (1-\tau)^{-\beta}f(\tau)d\tau, \qquad 0 < \beta < 1, \qquad 11$$

where Γ (•)is the Gamma function, D^{β}_{τ} denotes the material time-differentiation of fractional order. It is remarked that Eq. (10) includes Eq. (7) as a special case for β=1 and for β=0, α_1= 0 and β_3=0, we get the constitutive relationship for viscous fluid.

$$\mathbf{V} = u(y,t)\,\hat{i}, \qquad 12$$

where *u* and *i* are the velocity and unit vector in the direction of *x*-coordinate.

On using Eq. (12), Eq. of continuity (2) is identically satisfied and the momentum equation (1), with the help of (4), in the absence of pressure-gradient, yields

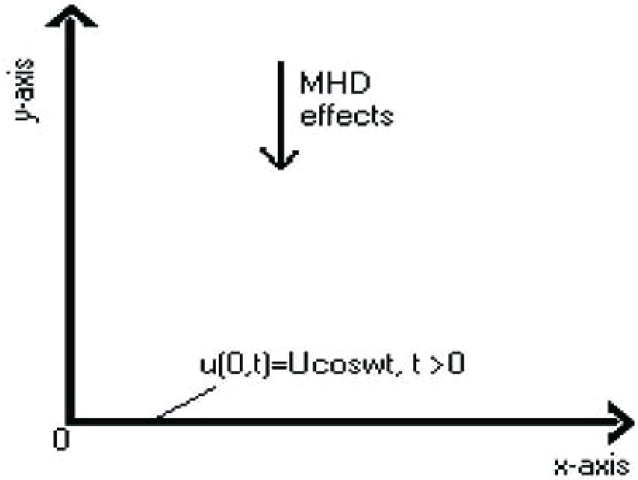

Figure - 1: Geometry of the Problem

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + \overline{\alpha}_1\frac{\partial^\beta}{\partial t^\beta}\frac{\partial^2 u}{\partial y^2} + \varepsilon\left[\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2\right] - Nu, \qquad 13$$

where

$$\nu = \frac{\mu}{\rho},\ \overline{\alpha}_1 = \frac{\alpha_1}{\rho},\ \varepsilon = \frac{6\beta_3}{\rho},\ N = \frac{\sigma \mathbf{B}_0^2(1+i\phi)}{\rho(1+\phi^2)}, \qquad 14$$

where $\phi = \varpi_e \tau_e$ is the Hall-parameter. The appropriate boundary conditions are:

$$u(0,t) = U\cos\omega t, \quad t > 0, \qquad 15a$$

$$u(y,t) \to 0, \quad \text{as} \quad y \to \infty, \qquad 15b$$

where U is the reference velocity and ϖ is the oscillating-frequency.

SOLUTION OF THE PROBLEM

For solution of the problem, we write [10]:

$$u(y,t;\varepsilon) = u_0(y,t) + \varepsilon u_1(y,t) + \cdots \qquad 16$$

in Eq. (13) and conditions (15) and, then equating like powers of ε, we obtain the following systems:

System of $O(\varepsilon^o)$

$$\frac{\partial u_0}{\partial t} = \nu\frac{\partial^2 u_0}{\partial y^2} + \overline{\alpha}_1\frac{\partial^\beta}{\partial t^\beta}\frac{\partial^2 u_0}{\partial y^2} - Nu_0, \qquad 17$$

$$u_0(0,t) = U\cos\omega t, \quad t > 0, \qquad 17a$$

$$u_0(y,t) \to 0 \text{ as } y \to \infty \qquad 17b$$

System of $O(\varepsilon)$

$$\frac{\partial u_1}{\partial t} = \nu\frac{\partial^2 u_1}{\partial y^2} + \overline{\alpha}_1\frac{\partial^\beta}{\partial t^\beta}\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_0}{\partial y^2}\left(\frac{\partial u_0}{\partial y}\right)^2 - Nu_1, \qquad 18$$

$$u_1(0,t) = 0, \qquad 18a$$

$$u_1(y,t) \to 0 \text{ as } y \to \infty. \qquad 18b$$

We solve the above systems by Fourier-transform method. For that we define the Fourier-transform as follows:

$$\psi(y,w) = \int_{-\infty}^{\infty} u(y,t)e^{-iwt}dt, \qquad 19a$$

$$u(y,t) = \int_{-\infty}^{\infty} \psi(y,w)e^{iwt}dw, \qquad 19b$$

in which w is the temporal frequency. The Fourier-transform of the fractional derivative D^{β}_{t} is defined as:

$$\int_{-\infty}^{\infty} D_t^{\beta}[u(y,t)]e^{-iwt}dt = (iw)^{\beta}\psi(y,w), \qquad 20$$

where

$$(iw)^{\beta} = |w|^{\beta}\exp[i\beta\pi/isignw] = |w|^{\beta}(\cos\beta\pi/2 + i\sin\beta\pi/2).$$

With the help of Eq. (19a), the transformed problem takes the following forms

$$\frac{d^2\psi_0}{dy^2} - \lambda^2\psi_0 = 0, \qquad 21$$

$$\psi_0(0,w) = U\pi[\delta(w-w_0)+\delta(w+w_0)], \qquad 21a$$

$$\psi_0(\infty,w) = 0, \qquad 21b$$

where

$$\lambda^2 = \frac{iw+N}{v+\alpha(iw)^{\beta}} = \frac{|w|^{1-\beta}\, i\mathrm{sign}w + N|w|^{-\beta}}{v\left[|w|^{-\beta} + \alpha_1/\mu\{\cos\beta\pi/2 + i\mathrm{sign}w\sin\beta\pi/2\}\right]}. \qquad 22$$

Solving Eq. (21) with conditions (21a) and (21b), then using Eq. (19a) were arrive at a result:

$$u_0(y,t) = \frac{U}{2}[e^{-\lambda_1 y}e^{iw_0 t} + e^{-\lambda_{-1}y}e^{-iw_0 t}], \qquad 23$$

where

$$\lambda_{\pm 1} = [\lambda]_{w=\pm w_0}.$$

To obtain the first-order solution, we substitute the zeroth order solution in (18) we get:

$$\frac{\partial u_1}{\partial t} = v\frac{\partial^2 u_1}{\partial y^2} + \alpha_1\frac{\partial^{\beta}}{\partial t^{\beta}}\frac{\partial^2 u_1}{\partial y^2} + \frac{U^3}{8}\begin{bmatrix} \lambda_1^4 e^{-3\lambda_1 y}e^{3iw_0 t} + \lambda_{-1}^4 e^{-3\lambda_{-1}y}e^{-3iw_0 t} \\ +(\lambda_1^2\lambda_{-1}^2 + 2\lambda_1^3\lambda_{-1})e^{-(2\lambda_1+\lambda_{-1})y}e^{iw_0 t} \\ +(\lambda_1^2\lambda_{-1}^2 + 2\lambda_{-1}^3\lambda_1)e^{-(2\lambda_{-1}+\lambda_1)y}e^{-iw_0 t} \end{bmatrix}. \qquad 24$$

The Eq. (24) and conditions (18a,b) after using Eq. (19a) become:

$$\frac{d^2\psi_1}{dy^2} - \lambda^2\psi_1 = c\begin{bmatrix} \lambda_1^4 e^{-3\lambda_1 y}\delta(w-3w_0) + \lambda_{-1}^4 e^{-3\lambda_{-1}y}\delta(w+3w_0) \\ ae^{-(2\lambda_1+\lambda_{-1})y}\delta(w-w_0) + be^{-(2\lambda_{-1}+\lambda_1)y}\delta(w+w_0) \end{bmatrix}, \qquad 25$$

$$\psi_1(0,w) = 0, \qquad 25a$$

$$\psi_1(\infty,w) = 0, \qquad 25b$$

where:

$$c = \frac{-U^3}{8\left[v+\alpha(iw)^{\beta}\right]},\ a = \lambda_1^2\lambda_{-1}^2 + 2\lambda_1^3\lambda_{-1},\ b = \lambda_1^2\lambda_{-1}^2 + 2\lambda_{-1}^3\lambda_1.$$

The solution of Eq. (25) satisfying the conditions (25a) and (25b) is of the following form

$$\psi_1(y,w) = c\begin{bmatrix} \frac{\lambda_1^4}{\lambda_2^2-\lambda^2}\delta(w-3w_0)\{e^{-\lambda y} - e^{-3\lambda_1 y}\} \\ +\frac{\lambda_{-1}^4}{\lambda_{-2}^2-\lambda^2}\delta(w+3w_0)\{e^{-\lambda y} - e^{-3\lambda_{-1}y}\} \\ \frac{a}{(2\lambda_1+\lambda_{-1})^2-\lambda^2}\delta(w-w_0)\{e^{-\lambda y} - e^{-(2\lambda_1+\lambda_{-1})y}\} \\ +\frac{b}{(2\lambda_{-1}+\lambda_1)^2-\lambda^2}\delta(w+w_0)\{e^{-\lambda y} - e^{-(2\lambda_{-1}+\lambda_1)y}\} \end{bmatrix}. \qquad 26$$

The inverse Fourier transform of above equation gives:

$$u_1(y,t) = \frac{U^3}{8[v+\alpha(iw)^{\beta}]}\begin{bmatrix} \frac{\lambda_1^4}{9\lambda_1^2-\lambda_2^2}\{e^{-\lambda_2 y} - e^{-3\lambda_1 y}\}e^{3iw_0 t} \\ +\frac{\lambda_{-1}^4}{9\lambda_{-1}^2-\lambda_{-2}^2}\{e^{-\lambda_{-2}y} - e^{-3\lambda_{-1}y}\}e^{-3iw_0 t} \\ +\frac{\lambda_1^2\lambda_{-1}^2+2\lambda_1^3\lambda_{-1}}{(2\lambda_1+\lambda_{-1})^2-\lambda_1^2}\left\{e^{-\lambda_1 y} - e^{-(2\lambda_1+\lambda_{-1})y}\right\}e^{iw_0 t} \\ +\frac{\lambda_1^2\lambda_{-1}^2+2\lambda_{-1}^3\lambda_1}{(2\lambda_{-1}+\lambda_1)^2-\lambda_{-1}^2}\left\{e^{-\lambda_{-1}y} - e^{-(2\lambda_{-1}+\lambda_1)y}\right\}e^{-iw_0 t} \end{bmatrix}, \qquad 27$$

in which

$$\lambda_{\pm 2} = \lambda/_{w=\pm 3w_0}.$$

CONCLUSIONS

The fractional calculus approach in the constitutive relationship model of a generalized third-grade fluid is taken into account. Unsteady uni-directional flow of the third-grade fluid with fractional derivative model is constructed. Using the Fourier-transform of the sequential fractional derivative, we obtain the perturbation solutions of the velocity-field. It is found that, in case of steady flow, the results for third-grade and generalized third-grade fluids are identical. The present analysis is more useful than the ordinary third-grade model for describing the properties of third-grade fluid.

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