ON PLANE FLOWS OF AN INCOMPRESSIBLE FLUID OF VARIABLE VISCOSITY

*Rana Khalid Naeem**

ABSTRACT

In this paper ,we indicate that the equations governing the motion of an incompressible fluid of variable viscosity for an arbitrary state equation possess an infinite set of solutions for the flows characterized by the equation $y = R(x) + C_1\Psi + C_2$ where Ψ is the streamfunction and R(x) the arbitrary function. The streamline patterns for the flows in unbounded and bounded domains are also presented for some forms of the function R(x).

Key words: Incompressible fluid of variable viscosity, plane flows , a class of plane flows, ,plane flows using von-Mises coordinates.

1. INTRODUCTION

The objective of this paper is to indicate that the equations describing the motion of an incompressible fluid of variable viscosity possess an infinite set of solutions for the flows characterized by the equation $y = R(x) + C_1 \Psi + C_2$ where Ψ is the streamfunction, R(x) is the arbitrary function, and C_1 ($\neq 0$) and C_2 are arbitrary constants .To achieve our objective, we treat x and Ψ as the independent variables and recast the flow equations in the von-Mises coordinates system (x , Ψ). von-Mises coordinates are used by many researchers [1-4] to obtain the numerical solutions of the flow equations for inviscid fluids.The paper is organized in the following fashion:In section (2), we transform the flow equations in the von-Mises system (x, Ψ).In section (3), we determine the exact solutions of the flow equations using the method of differentiation and integration [5,6] and present flow patterns for some forms of the function R(x) in unbounded domains and boundary value problems.

2. BASIC FLOW EQUATIONS

The non-dimensional equations describing the steady plane motion of an incompressible of fluid of variable viscosity μ and constant thermal conductivity κ are [7]

$$u_x + v_y = 0 \quad (1)$$

$$u u_x + v u_y = -p_x + [(2\mu u_x)_x + (\mu\{u_y + v_x\})_y]/Re \quad (2)$$

$$u v_x + v v_y = -p_y + [(2\mu v_y)_y + (\mu\{u_y + v_x\})_x]/Re \quad (3)$$

$$u T_x + v T_y = [T_{xx} + T_{yy}]/Re\,Pr + Ec\,[2\mu(u_x^2 + v_y^2) + \mu(v_x + u_y)^2]/Re \quad (4)$$

The meaning of various symbols used here are given in nomenclature. On introducing the vorticity ω and the total energy function L defined by:

$$\omega = v_x - u_y \quad (5)$$

$$L = p + (u^2 + v^2)/2 - 2\mu u_x/Re \quad (6)$$

the equations (1-4)can be rewritten as:

$$u_x + v_y = 0 \quad (7)$$

$$-v\omega = -L_x + [\mu\{u_y + v_x\}]_y/Re \quad (8)$$

$$u\omega = -L_y + [-(4\mu u_x)_y + (\mu\{u_y + v_x\})_x]/Re \quad (9)$$

$$u T_x + v T_y = [T_{xx} + T_{yy}]/Re\,Pr + Ec\,[2\mu(u_x^2 + v_y^2) + \mu(v_x + u_y)^2]/Re \quad (10)$$

In von-Mises coordinates (x , Ψ) , the velocity components u ,v and partial derivatives $(\bullet)_x$, $(\bullet)_y$ are given by

$$u = 1/y_\Psi \quad , \quad v = y_x/y_\Psi \quad (11)$$

$$(\bullet)_x = (\bullet)_x + (y_x/y_\Psi)(\bullet)_\Psi$$
$$(\bullet)_y = (1/y_\Psi)(\bullet)_\Psi \quad (12)$$

where in above " • " denotes any function. Equations (7-10) ,using Eqs. (11),(12) ,and $y = R(x) + C_1\Psi + C_2$,are replaced by

$$R'' = -C_1 L_\Psi - R' R'' \mu_\Psi/Re + C_1(\mu R'')_x/Re \quad (13)$$

$$-C_1 L_x + R'(\mu R'')_x/Re + (1 + R'^2)R''\mu_\Psi/C_1 Re = 0 \quad (14)$$

$$T_{xx} - R'' T_\Psi/C_1 - 2R' T_{x\Psi}/C_1 + (1 + R'^2)T_{\Psi\Psi}/C_1^2 - Re\,Pr\,T_x/C_1 = -Ec\,Pr\,\mu R''^2/C_1^2 \quad (15)$$

* Department of Mathematics, University of Karachi, Karachi – Pakistan. Email: rknwindsor@ yahoo.com

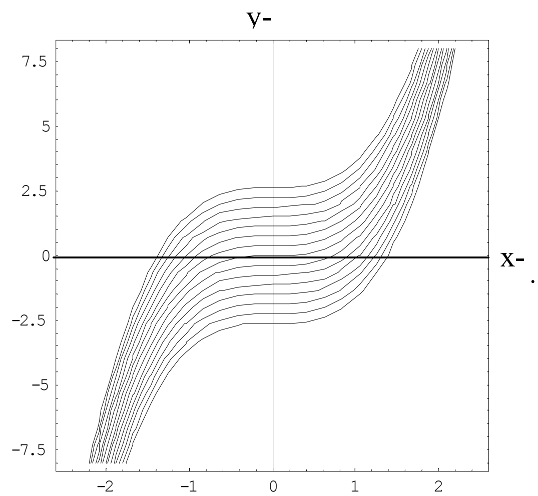

Figure – 1: Streamline pattern for $y-x^3$=constant (unbounded domain)

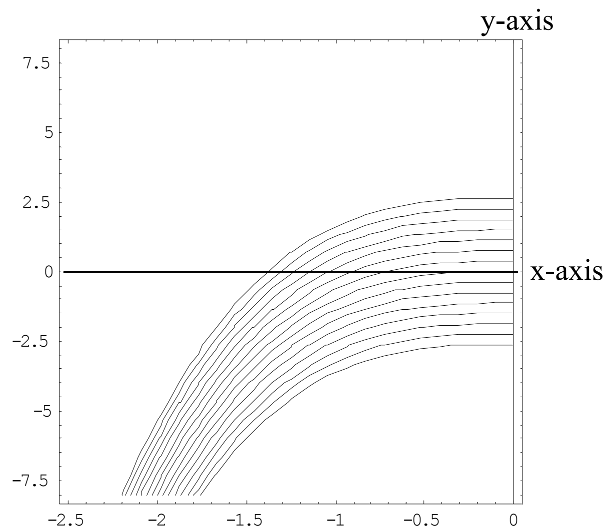

Figure – 2: Streamline pattern for $y-x^3$=constant for boundary value problem

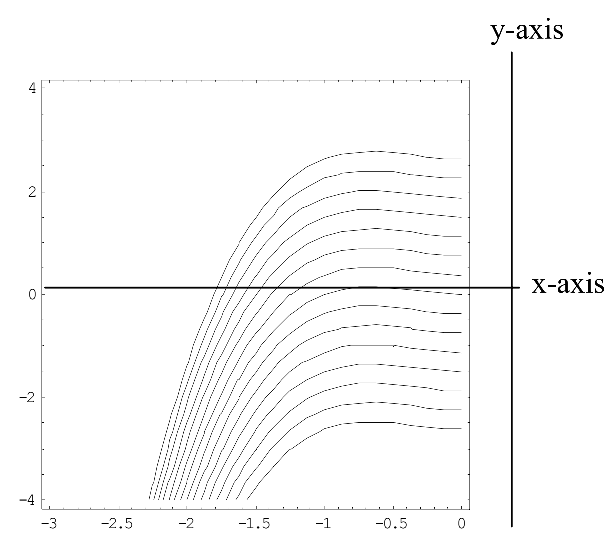

Figure-3: Streamlines pattern for $y-x^3-x^2$ = constant (unbounded domain)

3. SOLUTIONS

In this section we determine some solutions of the system of equations (13-15).

Equations (13) and (14), on using the integrability condition
$L_{xy} = L_{yx}$, yield

$$R'' \mu_{xx} - 2 R' R'' \mu_{x\Psi} / C_1 + 2 R''' \mu_x - [(R' R'')_x + R' R'''] \mu_\Psi / C_1 - R'' (1 + R'^2) \mu_{\Psi\Psi} / C_1^2 + R^{IV} \mu = \mathrm{Re}\, R''' / C_1 \quad (16)$$

Equation(16) is the equation which the function R(x) and viscosity μ must satisfy for the flows characterized by Eq.(11). Once a solution of this is determined, the function L and temperature T are determined from Eqs.(13–15). If we treat Eq.(16) as a partial differential equation in μ the coefficients suggests to assume μ of the form

$$\mu = M(x) + M_1(x)\, \nu \quad (17)$$

Equation (16), on utilizing Eq.(17), becomes

$$R'' [M'' + M_1'' \nu (\Psi)] - 2 R' R'' M_1' + 2 R''' [M' + M_1' \nu (\Psi)] - [(R' R'')_x + R' R'''] M_1 / C_1 + R^{IV} (M + M_1 \nu (\Psi)) = \mathrm{Re}\, R''' / C_1 \quad (18)$$

where

$$\nu(\Psi) = C_1 \Psi + C_2$$

Since x and Ψ are independent variables, therefore, Eq.(18) holds for all x and Ψ provided:

$$R'' M_1'' + 2 R''' M_1' + R^{IV} M_1 = 0 \quad (19)$$

and

$$R'' M'' - 2 R' R'' M_1' + 2R''' M' - [(R' R'')_x + R' R'''] M_1 / C_1 + R^{IV} M = \mathrm{Re}\, R''' / C_1 \quad (20)$$

The solution of Eq.(19), for $R'' \neq 0$, is

$$M_1 = (C_3 x + C_4) / R'' \quad (21)$$

where C_3 (≠ 0) and C_4 are arbitrary constants .

Substituting Eq.(21) in Eq.(20) and integrating twice the resulting equation, we obtain:

$$M(x) = \left(\int [\int N(x)\, dx + C_5]\, dx + C_6 \right) / R'' \quad (22)$$

where

$$N(x) = \mathrm{Re}\, R''' / C_1 + 2 R' R'' M_1' + [(R' R'')_x + R' R'''] M_1 / C_1 \quad (23)$$

and C_5, C_6 are arbitrary constants. Since in Eqs.(21) and (22), the function R(x) is arbitrary, we can generate a large number of expressions for viscosity μ .

The temperature Eq.(15), for viscosity μ defined in Eq.(17), becomes

$$T_{xx} - \mathrm{Re}\,\mathrm{Pr}\, T_x / C_1 - 2 R' T_{x\Psi} / C_1 - R'' T_\Psi / C_1 + (1 + R'^2) T_{\Psi\Psi} / C_1^2 = - \mathrm{Ec}\,\mathrm{Pr}\, R''^2 (M + M_1 \nu) C_1^2 \quad (24)$$

We seek a solution of Eq. (24) of the form

$$T = T_1 (x) + T_2 (x) \nu + T_3 (x) \nu^2 + \zeta (\nu) \quad (25)$$

Eq.(15) , on using Eq.(25). , yields

$$N_1 (x) + N_2 (x) \nu + N_3 (x) \nu^2 + (1 + R'^2) \zeta'' - R'' \zeta' = 0 \quad (26)$$

where the functions $N_1(x)$, $N_2(x)$, $N_3(x)$ are given in appendix A.

Differentiating Eq.(26) w. r .t ‘ν ‘ , we get:

$$N_2 + 2 N_3 \nu + (1 + R'^2) \zeta''' - R'' \zeta'' = 0 \quad (27)$$

Differentiating Eq.(27) w. r .t ‘ ν ‘ , we get:

$$2 N_3 + (1 + R'^2) \zeta^{IV} - R'' \zeta''' = 0 \quad (28)$$

Differentiating Eq.(28) w. r .t ‘ ν ‘ , we obtain:

$$(1 + R'^2) \zeta^{V} / R'' - \zeta^{IV} = 0 \quad (29)$$

On differentiating Eq.(29) w. r .t ‘ x ‘ , we get:

$$[(1 + R'^2) / R'']' \zeta^{V} = 0 \quad (30)$$

This holds for all x and ν (Ψ) provided:

(i) $\zeta^{V} = 0$, $[(1 + R'^2) / R'']' \neq 0$.

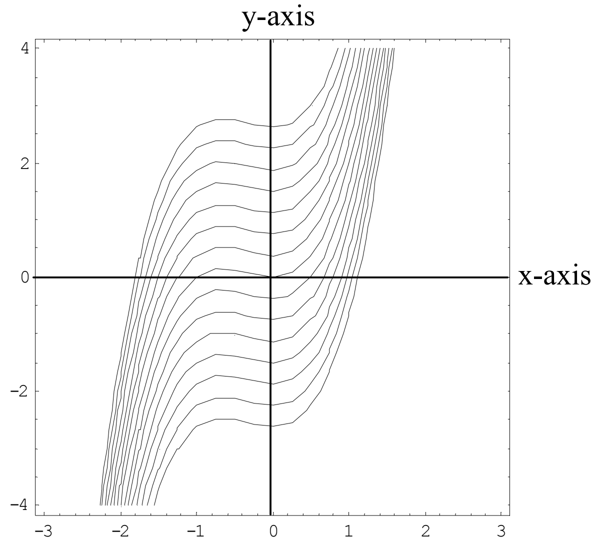

Figure-4: Streamlines pattern for $y-x^3-x^2$ = constant for boundary value problem

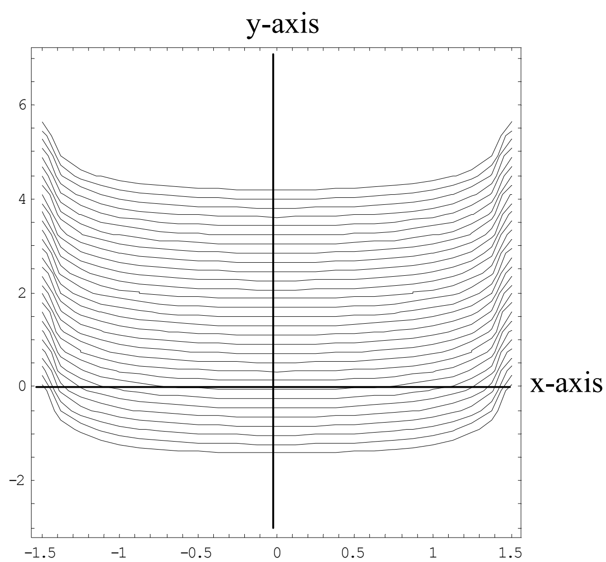

Figure-5: Streamlines pattern for y – log[Secx + 3] = constant for unbounded domain

(ii) $[(1+R'^2)/R'']' = 0$, $\zeta^V \neq 0$.

We consider these two cases separately.

Case (i) When $\zeta^V = 0$, we get

$$\zeta(\nu) = C_7 \nu^4/24 + C_8 \nu^3/6 + C_9 \nu^2/2 + C_{10}\nu + C_{11} \quad (31)$$

where C7 ,C8, C9, C10 ,C11 are constants . Substituting Eq. (31) in Eq.(29), we get

$$C_7 = 0 \quad (32)$$

Equations (32) and (28) , yield

$$2N_3 - R'' C_8 = 0 \quad (33)$$

The solution of Eq (33), utilizing the expression for $N_3(x)$, is

$$T_3 = \left[\int (C_8 R'/2 + C_{12}) \exp(-Re\,Pr\,x/C_1)\,dx + C_{13}\right] \exp(Re\,Pr\,x/C_1) \quad (34)$$

where C12, C13, are non-zero arbitrary constants.

Equation (27), on using Eqs. (31-33), yields

$$N_2 = C_9 R'' - C_8(1+R'^2) \quad (35)$$

On substituting expression for $N_2(x)$ in Eq. (35) and integrating twice the resulting differential equation , we get

$$T_2 = \int \exp(Re\,Pr\,x/C_1)\left[\int \exp(-Re\,Pr\,x/C_1)\, N_4(x)\,dx + C_{14}\right] dx + C_{15} \quad (36)$$

where

$$N_4(x) = C_9 R'' - C_8(1+R'^2) + 2R'' T_3 + 4R' T_{3x} - Ec\,Pr\,R''^2 M_1 / C_1^2 \quad (37)$$

Finally, utilizing the above results in Eq.(26) and the expression for $N_1(x)$, we obtain

$$T_1 = \int \exp(Re\,Pr\,x/C_1)\left[\int N_5 \exp(-Re\,Pr\,x/C_1)\,dx + C_{16}\right] dx + C_{17} \quad (38)$$

where

$$N_5 = C_{10} R'' - C_9(1+R'^2) + R'' T_2/C_1 + 2R' T_{2x} - 2(1+R'^2)T_3 - Ec\,Pr\,R''^2 M/C_1^2 \quad (39)$$

We note that the function R(x) is still arbitrary and, therefore, we can construct a large number of solutions to the flow equations. In Eqs.(36-38) , C_{14}, C_{15}, C_{16}, C_{17} are all non-zero arbitrary constants.

Case (ii) When $[(1+R'^2)/R'']' = 0$, then

$$R = [\ln\sec(x/C_{18} + C_{19})]/C_{18} + C_{20} \quad (40)$$

where C_{18} ($\neq 0$), C_{19}, C_{20} are arbitrary constants .

The expression for the function R must satisfy the Eq. (29) and therefore we get:

$$C_{18}\zeta^V - \zeta^{IV} = 0 \quad (41)$$

The solution of Eq.(41) is:

$$\zeta = \exp(\nu/C_{18})\left\{\int\left[C_{21}\nu^3/6 + C_{22}\nu^2/2 + C_{23}\nu + C_{24}\right]\exp(-\nu/C_{18})\,d\nu + C_{25}\right\} \quad (42)$$

where C_{21}, C_{22}, C_{23}, C_{24}, C_{25} are all non-zero constants.

Equation (28) , utilizing Eqs.(40) and (41), yields

$$N_3 = -C_{21} C_{18} R''/2 \quad (43)$$

Equation (27),employing Eqs.(40-43) , yields

$$N_2 = -C_{22} C_{18} R'' \quad (44)$$

Substituting the expressions of $\zeta(\Psi)$, N_2, N_3 in Eq.(26), we get:

$$N_1 = -C_{18} C_{23} R'' \quad (45)$$

On using the definitions of N_1, N_2, N_3 from appendix A and after integration , we obtain

$$T_1 = Z_3\left[\int Z_3^{-1}\left\{\int Z_2\,dx + C_{31}\right\}dx\right] + C_{32} Z_3 \quad (46)$$

$$T_2 = Z_3\left[\int Z_3^{-1}\left\{\int Z_1\,dx + C_{29}\right\}dx + C_{30}\right] \quad (47)$$

$$T_3 = -C_{21} C_{18}\left[\int Z_3\left\{\int R' Z_3^{-1}\,dx\right\}dx\right]/2 - C_1^2 C_{26} Z_3/Re^2 Pr^2 + C_1 C_{27} Z_3/Re\,Pr + C_{28} \quad (48)$$

In Eqs.(46 –47) ,

$$Z_1 = -C_{18} C_{22} R'' + 2R'' T_3 + 4R' T_{3x}$$

$- Ec\, Pr\, R'' M_1(x) / C_1^2$ (49)

$Z_2 = -C_{18} C_{23} R'' + R'' T_2 / C_1 + 2 R' T_{2x} - 2(1 + R'^2) T_3 - Ec\, Pr\, R''^2 M(x) / C_1^2$ (50)

$Z_3 = \exp(Re\, Pr\, x / C_1)$ (51)

The solution of Eq.(16) can further be extended by assuming:

$\mu = M(x) + M_1(x)\nu + M_2(x)\nu^2$ (52)

Substituting Eq.(52) in Eq.(16) , we get:

$X_1(x) + X_2(x)\nu + X_3(x)\nu^2 = 0$ (53)

where the functions $X_1(x), X_2(x), X_3(x)$, are given in appendix A.

The solution of Eq.(53), following the same procedure as that of Eq.(18), is given by:

$M_2 = (a_1 x + a_2) / R''$, $R'' \neq 0$ (54)

$M_1 = \{\int [\int L_1(x)\, dx + a_3]\, dx + a_4\} / R''$ (55)

$M = \{\int [\int L_2(x)\, dx + a_3]\, dx + a_5\} / R''$ (56)

where

$L_1(x) = 4 R' R'' M_2' + 2[(R' R'')_x + (R' R''')] M_2$ (57)

$L_2(x) = 2 R' R'' M_1' + [(R' R'')_x + R' R'''] M_1' + 2 R'' (1 + R'^2) M_2' + Re\, R''' / C_1$ (58)

and a_1, a_2, a_3, a_4, a_5 are all non-zero arbitrary constants. We note that the function R(x) in Eq.(54– 56) is arbitrary and therefore we can construct a large number of expressions for viscosity μ and the velocity components u , v .

The form for μ in Eq. (52), suggests to seek a solution of Eq.(15) in the form:

$T = T_1(x) + T_2(x)\nu + T_3(x)\nu^2 + T_4(x)\nu^3 + \zeta(\nu)$ (59)

This, on substituting in Eq.(15), we obtain:

$N_6(x) + N_7(x)\nu + N_8(x)\nu^2 + N_9(x)\nu^3 + (1 + R'^2)\zeta'' - R''\zeta' = 0$ (60)

where $N_6(x)$, $N_7(x)$, $N_8(x)$, $N_9(x)$ are given in appendix B.

The solution of Eq.(60) can easily be determined in the same manner as that of Eq.(26) . We mention that the function R(x) in the solution of Eq.(60) also remains arbitrary, and, therefore, we can construct a large number of solutions to the flow equations.

Finally, for the sake of completeness, we consider some forms of the function R(x) and the boundary value problems .When $R(x) = x^3$, the flow pattern in unbounded domain is sketched in Figure 1. The Figure 2 represents the flow pattern for the fluid impinge on a porous wall
x = 0 , satisfying boundary conditions u(0,y) = 1 , v(0, y) = 0. For $R(x) = x^3 + x^2$, the streamline pattern in unbounded domain and for the boundary value problem same as for $R(x) = x^3$ are given in figures (3) and (4) ,respectively. The streamline pattern for the function R(x) defined by Eq.(40) for some values of C_{18} (≠ 0), C_{19}, C_{20} is sketched in figure (5).

4. CONCLUSIONS

It is indicated that the equations governing the steady plane motion of an incompressible fluid of variable viscosity for an arbitrary state equation admit an infinite set of solutions for a class of flows characterized by $y = R(x) + C_1\Psi + C_2$ .Some flow patterns for some forms of the arbitrary function R(x) are also presented in unbounded and bounded domains.

REFERENCES

- R.M.Barron . Computation of interminable potential flow using von Mises coordinates, Math. Comput. Simulation, 31(3), 177,1989.
- R.M.Barron and R.K.Naeem Numerical solution of transonic flows on a streamfunction coordinate system. Int. J. Numer. Methods fluids, 9(10), 1183,1989.
- R.K.Naeem and R.M.Barron . Transonic Computation on a natural grid , AIAA J , 28(10), 836,1990.
- R.M.Barron and R.K.Naeem . 2-D. transonic calculations on a flow based grid system , Mathematics and Computers on sanitation North Holland, 33, 65,1991.
- R.K.Naeem and S.Afreen .A class of exact motion of a Conduction incompressible fluid of variable viscosity in the presence of a transverse magnetic field,30(12), 99, 2002.
- R.K.Naeem and S.S.Zia. some exact solutions to second grade fluid flow

equations using complex variable technique, Int. Journal of Applied Mechanics and Engineering, 8(1), 71, 2003.

- R.K.Naeem and S.A.Nadeem. Study of plane steady flows of on incompressible fluid of variable viscosity using Martin's Method- Applied Mechanics and Engineering , 1(3), 397,1996.

NOMENCLATURE

u , v	–	velocity components.
x , y	–	cartesian coordinates.
p	–	pressure of fluid.
μ	–	viscosity coefficient.
T	–	temperature
Re	–	Reynolds number.
Pr	–	Prandtl number.
Ec	–	Eckert number.
Ψ	–	streamfunction.
E, F, G	–	coefficients of the fundamental elements.
W, J	–	Jacobin of transformation.
q	–	speed of the fluid
ω	–	vorticity
L	–	energy function.
α	–	angle of inclination.
Γ_{11}^{2}, Γ_{12}^{2}	–	christopher symbols.
A , M ,R , M_1, ζ , N , N_1 , N_2 , N_3 , N_4 ,N_5 ,N_6 ,N_7 , T_1,T_2 ,T_3 ,T_4 , Z_1 , Z_2 , Z_3, X_1, X_2 , X_3	–	functions

superscripts

′ ,″ ,‴,V, 1V	–	differentiation w .r .t. argument.

subscripts

x, y	–	differentiation w .r .t. cartesian coordinates.
Ψ	–	differentiation w.r.t Ψ coordinate.

APPENDIX - A

$N_1(x) = T_{1xx} - R^{''} T_2 / C_1 - 2 R^{'} T_{2x} + 2 (1 + R^{'2}) T_3 - Re\, Pr\, T_{1x} / C_1$

$+ Ec\, Pr\, R^{''2} M / C_1^{\,2}$

$N_2(x) = T_{2xx} - 2 R^{''} T_3 - Re\, Pr\, T_{2x} / C_1 + Ec\, Pr\, R^{''2} M_1 / C_1^{\,2}$

$N_3(x) = T_{3xx} - Re\, Pr\, T_{3x} / C_1$

$X_1 (x) = R^{''} M^{''} - 2 R^{'} R^{''} M_1^{'} + 2 R^{'''} M^{'} - [(R^{'} R^{''})_x + R^{'} R^{'''}] M_1^{'}$

$X_2 (x) = R^{''} M_1^{''} - 4 R^{'} R^{''} M_2^{'} + 2 R^{'''} M_1^{'} - 2 [(R^{'} R^{''})_x + R^{'} R^{'''}] M_2^{'}$

$+ R^{1V} M_1$

$X_3(x) = R^{''} M_2^{''} + 2 R^{'''} M_2^{'} + R^{1V} M_2$

APPENDIX - B

$N_6 (x) = T_{1xx} - Re\, Pr\, T_{1x} / C_1 - 2 R^{'} T_{2x} - R^{''} T_{2x} + 2 (1 + R^{'2}) T_{3x}$

$+ Ec\, Pr\, R^{''2} / C_1^{\,2}$

$N_7 (x) = T_{2xx} - Re\, Pr\, T_{2x} / C_1 - 4 R^{'} T_{3x} - 2 R^{''} T_{3x}$

$+ 6 (1 + R^{'2}) T_{4x} \quad + Ec\, Pr\, R^{''2} M_1 / C_1^{\,2}$

$N_8 (x) = T_{3xx} - Re\, Pr\, T_{3x} / C_1 - 6 R^{'} T_{4x} - 3 R^{''} T_{4x}$

$+ Ec\, Pr\, R^{''2} M_2 / C_1^{\,2}$

$N_9 (x) = T_{4xx} - Re\, Pr\, T_{4x} / C_1$