SOME NEW EXACT SOLUTIONS OF EQUATIONS OF MOTION OF A COMPRESSIBLE FLUID OF CONSTANT VISCOSITY, VIA ONE-PARAMETER GROUP

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ABSTRACT

Some new exact solutions for the equations of motion of a compressible fluid of constant viscosity are determined employing one parameter group of transformations for an arbitrary state equation. All the exact solutions contain arbitrary functions/ function and this arbitrariness of functions/ function enables us to construct a large number of solutions to flow equations. The streamline patterns for some solutions are also given.

Keywords: One parameter group; Navier-Stroke's equations; Viscous compressible fluid; Steady plane flows; Rotational flows.

NOMENCLATURE

u, v	=	Velocity Components
p	=	Pressure
ρ	=	Density of fluid
μ	=	Viscosity of fluid
k	=	Coefficient of thermal conductivity
T	=	Temperature
e	=	Internal energy
Ψ	=	The streamfunction
x, y	=	Cartesian co-ordinates
Φ_j	=	set of functions
x_i	=	Independent variables
y_j	=	Dependent variables
X, H, R, f_i, A, B, C, D, J, G	=	functions
	=	Universal gas constant
C_v	=	Specific heat at constant volume
Superscript ′ ′′ ′′′	=	Differentiation w.r.t. argument
Subscript x, y	=	Differentiation w.r.t. Cartesian co-ordinates

1. INTRODUCTION

The objective of this paper is to indicate some new exact solutions of equations governing the motion of a compressible thermally conducting fluid of constant viscosity using one parameter group of transformations for an arbitrary state equation. The paper is organised in the following fashion: In section 2, we consider the flow equations and transform them into a new system of equations using one parameter group of transformations. In section 3, we determine some exact solutions of new system of equations for an arbitrary state equation.

2. BASIC EQUATIONS

The steady plane flow of a compressible fluid of constant viscosity is governed by the system of four partial differential equations:

$$(\rho u)_x + (\rho v)_y = 0$$

$$\rho u u_x + \rho v u_y = -p_x + \mu \nabla^2 u$$

$$\rho u v_x + \rho v v_y = -p_y + \mu \nabla^2 v \qquad (1)$$

$$2\mu\, [u^2_x + v^2_y + 1/2\, u^2_y + u_y v_x + 1/2\, v^2_x]$$

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$$-2\mu/3\,[u^2_x + 2u_x v_y + v^2_y] + K\,[T_{xx} + T_{yy}] = u e_x + v\,e_y + p(u_x + v_y)$$

where u, v are the velocity components, p the pressure, ρ the density, μ the viscosity, K the constant coefficient of thermal conductivity, T the temperature and e the internal energy.
On introducing

$$\rho u = \psi_y$$
$$\rho v = \psi_x \qquad (2)$$

the system (1) is replaced by the following system:

$$\psi_y R\psi_{xy} + \psi^2_y R_x - R\psi_x\psi_{yy} - \psi_x\psi_y R_y = -p^*_x + \mu(R\psi_x)_{xy} + \mu(R\psi_y)_{yy}$$
$$-R\psi_y\psi_{xx} - R_x\psi_y\psi_x + \psi^2_x R_y + R\psi_x\psi_{xy} = -p^*_y - \mu(R\psi_x)_{xx} - \mu\,(R\psi_y)_{xy}$$
$$4\mu/3\,\{(R\psi_y)_x\}^2 + 4\mu/3\,\{(R\psi_x)_y\}^2 + \mu\{(R\psi_y)_y\}^2 \qquad (3)$$
$$-2\mu(R\psi_y)_y\,(R\psi_x)_x + \mu\{R\psi_x)_x\}^2 + 4\mu/3\,(R\psi_y)_x(R\psi_x)_y + K(T_{xx}+T_{yy})$$
$$= \psi_y e_x - \psi_x e_y + p\{(R\psi_y)_x - (R\psi_x)_y\}$$

of three equations in five unknown functions ψ, p, R, T, e as functions of x and y. In above, ψ is the stream function and the functions R and p* are given by

$$R = 1/\rho$$
$$p^* = p - \frac{4\mu}{3}\left\{\left(R\psi_y\right)_x - \left(R\psi_x\right)_y\right\} \qquad (4)$$

Once a solution of system of equations (3) is determined, the density ρ is determined from equation (4).

We now transform the system of partial differential equations (3) into a new system of ordinary differential equations using one-parameter group of transformations. We give here only one parameter group Γ_1 and its invariants that are utilised to obtain exact solutions of system (3). For details of one parameter group theory reader is referred to Ibragimov (1983), oliver (1986). Ovsiannikov (1982), Naeem (1994).

If Γ_1 is a group consisting of a set of transformations defined by

$$\bar{x} = a^n x, \qquad \bar{y} = a^m y, \qquad \bar{\psi} = a^j \psi \qquad \bar{p} = a^p p$$
$$\bar{R} = a^r R, \qquad \bar{T} = a^t T \qquad \bar{e} = a^j e$$

with parameter $a \neq 0$, Then the invariants for Γ_1 for system (3) are

$$\Psi = A(\theta) \qquad R = x^{\alpha_1}B(\theta) \qquad p = x^{\alpha_2}C(\theta)$$
$$T = x^{\alpha_3}D(\theta) \qquad e = x^{\alpha_4}E(\theta) \qquad \theta = y/x$$

wherein

$$\alpha_1 = r/n,\ \ \alpha_2 = \alpha_1 - 2 = p/n,\ \ \alpha_3 = t/n = 2\alpha_1 - 2,\ \ \alpha_4 = j/n = 2\alpha_1 - 2 \qquad (6)$$

The invariants (5) of Γ_1 transform system (3) into the following system of ordinary differential equations:

$$(\alpha_1 - 1)B\acute{A}^2 = -\alpha_2 C + \theta C' + \mu[(1-\alpha_1)H' + (\theta H')' + (B\acute{A})''] \qquad 7.1$$
$$(\alpha_1 - 1)\theta B\acute{A}^2 = -C' - \mu[(1-\alpha_1)\,(\alpha_1 - 2)H + (2\alpha_1 - 3)\theta H' - \theta(\theta H')']$$
$$-\mu[(\alpha_1 - 2)\,(B\acute{A})' - \theta(B\acute{A})''] \qquad 7.2$$
$$4\mu/3\,[(\alpha_1 - 1)\,B\,\acute{A} - \theta(B\acute{A})']^2 + 4\mu/3\,[H']^2 + \mu[B\acute{A})']^2$$
$$-2\mu\,(B\acute{A})'[1-\alpha_1)H + \theta H'] + \mu[1-\alpha_1)H + \theta H']^2$$

$+ 4\mu/3\, H'[(1-\alpha_1)BA' + \theta(BÁ)'] + K[\alpha_3(\alpha_3 - 1)D - (2\alpha_3 - 2)\,\theta D'$
$+\theta^2 D'' + D''] = \alpha_1 BÁM + \alpha_4 EÁ$ 7.3

for the five unknown functions A, B, C, D, E of θ. In system (7)

$$H = \theta BÁ$$

and

$$C = M - 4\mu/3\, \alpha_1 BÁ$$

3. SOLUTIONS

In this section, we determine the solutions of the system (7).
When $\alpha_1 \neq 1$, equations (7.1–7.2) give:

$$\begin{aligned}(\alpha_1 - 1)Á\, J = &-\alpha_2 C + (1-\alpha_1)\theta^2 Á\, J + \mu[(1+\theta^2)J'' + (4-\alpha_1)\theta J' \\ &+ (2-\alpha_1)J + (\alpha_1^2 - 5\alpha_1 + 6)\theta^2 J \\ &+ \{(6-2\alpha_1)\theta^3 + (2-\alpha_1)\theta\}J' + \theta^2(1-\theta^2)J'']\end{aligned} \quad (8)$$

wherein

$$J = BÁ$$

Equations (7.2) and (8) imply that:

$$\begin{aligned}(1-\alpha_1)J\{u+\theta^2)A'' + (4-\alpha_1)\theta Á\} + (1-\alpha_1)(1+\theta^2)ÁJ' \\ = [\mu(\alpha_1 - 2)(\alpha_1 - 3) + (\alpha_1 - 4)\theta]J \\ -[\mu(\alpha_1 - 4)(\alpha_1 - 3) + 3\mu(\alpha_1 - 3)(\alpha_1 - 4)\theta^2]J' \\ -[\mu(1+\theta^2)^2 J''' - 3\mu(4-\alpha_1)\theta(1+\theta^2)J''\end{aligned} \quad (9)$$

The solution of equation (9), for given J, is:

$$A = \int \frac{(1+\theta^2)^{\frac{\alpha_1 - 4}{2}}}{J} L(\theta)\, d\theta + a_2 \quad (10)$$

wherein

$$L(\theta) = \int \left[\frac{1}{(1-\alpha_1)} (1+\theta^2)^{\frac{2-\alpha_1}{2}} \text{ R.H.S. of eq. (9) } d\theta \right] + a_1$$

and a_1, a_2 are constants of integration.

The functions $B(\theta)$ and $C(\theta)$ are obtained from:

$$B = J/Á \quad (11)$$

$$\begin{aligned}C = 1/\alpha_2[(1-\alpha_1)ÁJ + (1-\alpha_1)\theta^2 ÁJ + \mu\{(1+\theta^2)J'' + (4-\alpha_1)\theta J' \\ + (2-\alpha_1)J + (\alpha_1^2 - 5\alpha_1 + 6)\theta^2 J + [(6-2\alpha_1)\theta^3 + (2-\alpha_1)\theta]J' \\ + \theta^2(1+\theta^2)J''\}]\end{aligned} \quad (12)$$

The equation (7.3) gives

$$E = 1/\alpha_4 Á\, [-\alpha_1 JC + \text{L.H.S. of (7.3)}] \quad (13)$$

In the solution given by equation (10 -13) of system (7), the functions $J(\theta)$ and $D(\theta)$ are arbitrary. This arbitrariness of functions J and D enables us to construct a large number of solutions to the flow equations in system (1) for an arbitrary state equation.

In above solution, the arbitrary function $D(\theta)$, however, can be determined if the state equation is known. For example, when the state equation is, $p = \rho GT$, we find

$$D(\theta) = B(\theta)\, M(\theta)/G$$

where G is the universal gas constant

II. When $\alpha_1=1$, the system (7) becomes

$$C + \theta C' + \mu[(\theta H')' + J''] = 0 \qquad (14.1)$$

$$-C' + \mu[(\theta^2 H')' + (\theta J')'] = 0 \qquad (14.2)$$

$$-2/3\, \mu\theta J' H' + \mu(4/3+\theta^2)H'^2 + \mu(4/3\, \theta^2 + 1)J'^2 + K[(1+\theta^2)D']' = MJ \qquad (14.3)$$

The solution of system (14) is

$$C = \mu[\theta^2 H' + \theta J'] + a_3$$

$$J = \frac{a_3 + a_4\mu\theta}{1+\theta^2} + \frac{a_5}{(1+\theta^2)^{1/2}}$$

$$D = \int x_2(\theta)d\theta + m_1 \tan^{-1}\theta + m_2$$

wherein

$$x_1(\theta) = \frac{1}{K}\left\{JM + \frac{2}{3}\mu\theta J'H' - \mu\left(\frac{4}{3}+\theta^2\right)H'' - \mu\left(\frac{4}{3}\theta^2+1\right)J''\right\} \qquad (16)$$

$$x_2(\theta) = \frac{1}{1+\theta^2}\int x_1(\theta)\,d\theta \qquad (17)$$

and m_1 , m_2, are arbitrary constants

We note that when $\alpha_1=1$, $\alpha_4=0$ and therefore the term $\alpha_4 E\acute{A}$ in equation (7.3) disappears and hence the function $E(\theta)$ becomes arbitrary.

In the above solution

$$B\acute{A} = a_3 + a_4\mu\theta/1+\theta^2 \;+\; a_5/(1+\theta^2)^{1/2} \qquad (18)$$

which means that one of the functions B or Á is arbitrary.

The solution (15) is for an arbitrary state equation and contains arbitrary functions $B(\theta)$ (or A'), $E(\theta)$ and this enables us to generate a large number of solutions to the flow equations. For ideal gases, the function $B(\theta)$ can be easily determined from $B(\theta) = GD(\theta)/M(\theta)$. However, the function $E(\theta)$still remains arbitrary. For polytropic gases the function $E(\theta)$ is given by $E(\theta) = C_v D(\theta)$,where C_v is the specific heat at constant volume.

III. When $\alpha_1=2$, the system (7) gives

$$J\acute{A} = \theta C' + \mu[(1+\theta^2)J']' \qquad (19)$$

$$\theta J\acute{A} = -C' + \mu[\theta^2(\theta J)'' + \theta J''] \qquad (20)$$

$$4\mu J^2/3 + \mu(1+\theta^2)^2 J'^2 + K[(1+\theta^2)D'' - 2\theta D' + 2D] = 2JC + E\acute{A} \qquad (21)$$

The equations (19–20), on eliminating C', yield

$\mu(1+\theta^2)J'' + 2\theta\mu J' - \acute{A}J = 0$ (22)

The integration of equation (20) yields
$C = \int[-\theta J\acute{A} + \mu\{\theta^2(\theta J)'' + \theta J'\}']d\theta + \lambda^*$
wherein λ^* is a constant of integration

The solution of equation (21) is
$D = \theta\int(1+\theta^2)/\theta^2 \; X(\theta)\,d\theta + \lambda_1^*\theta$
wherein
$X(\theta) = \int \theta^2/K(1+\theta^2)\,[2JC + 2E\acute{A} - 4\mu/3\,J^2 - \mu(1+\theta^2)^2 J'^2]\,d\theta + \lambda_2^*$
and λ_1, λ_2^* are constants and the function $E(\theta)$ is an arbitrary function. The solution of equation (22) is determined as follows:

We know from the theory of ordinary differential equations that equation
$\Phi''(\theta) + a_1(\theta)\Phi'(\theta) + a_2(\theta)\Phi = 0$

can be put in the normal form
$\Phi^{*''} + (a_2(\theta) - 1/2\,a_1'(\theta) - 1/4\,a_1^2(\theta))\Phi^* = 0$ (23)

by substituting $\Phi(\theta) = \Phi^*(\theta)\,e^{-1/2\int a_1(\theta)\,d\theta}$ (24)

The equation (22), on utilising equations (23-24), yields

$\Phi^{*''} + (-\acute{A}/\mu(1+\theta^2) - 1/(1+\theta^2)^2)\Phi^* = 0$ (25)
wherein
$J = \Phi^*(1+\theta^2)^{-1/2}$ (26)

The solution of equation (25) can easily be determined by appropriately choosing the coefficient of Φ^* i.e.,

$-\acute{A}/\mu(1+\theta^2) - 1/(1+\theta^2)^2$ (27)
Let us now consider some of the choices for which the solution of equation (25)can easily be determined
(i) If we take
$-\acute{A}/\mu(1+\theta^2) - 1/(1+\theta^2)^2 = \lambda/\theta^2$ (28)
then
$A = -\mu\tan^{-1}\theta + \lambda\mu/\theta - \lambda\mu\theta + d_1$ (29)
and

$$\Phi^* = \begin{cases} d_2\theta^{\frac{1+\sqrt{1-4\lambda}}{2}} + d_3\theta^{\frac{1-\sqrt{1-4\lambda}}{2}}, & 4\lambda < 1 \\ \theta^{1/2}\left(d_4\cos\sqrt{4\lambda-1}\,\ln\theta + d_5\sin\sqrt{4\lambda-1}\,\ln\theta\right), & 4\lambda > 1 \\ (d_6 + d_7\ln\theta)\theta^{1/2}, & 4\lambda = 1 \end{cases} \quad (30)$$

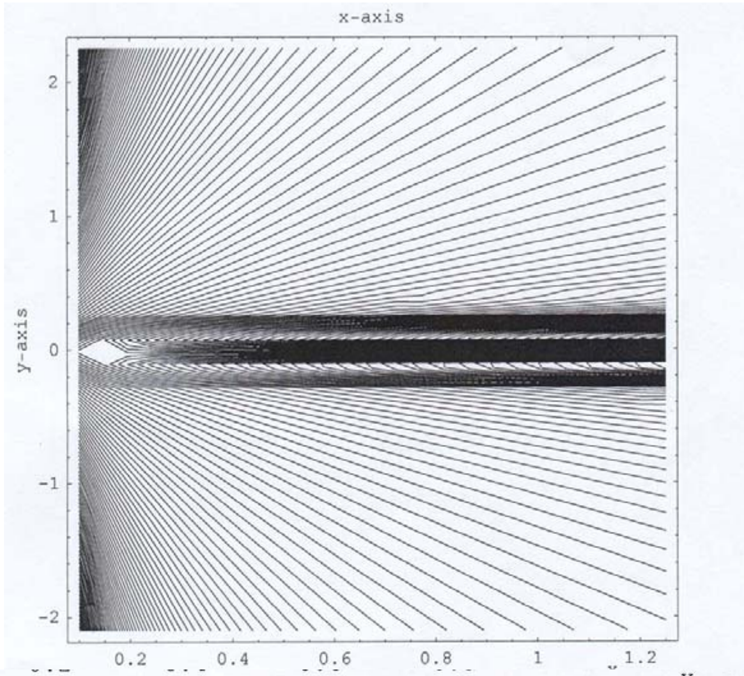

Figure-1: Streamlines pattren for $12+12\mathrm{ArcTan}[y/x]-6(\frac{x}{y})+6(y/x)=0$

where $\lambda \neq 0$, d_1, d_2, d_3, d_4, d_5, d_6, d_7 are all non-zero constants. The stream function is $\psi = A(\theta) = A(y/x) = -\mu\tan^{-1}(y/x) + \lambda\mu(x/y) - \lambda\mu(y/x) + d_1$. The streamlines pattern is given in Figure 1.

Equation (26), on using equation (30), gives

$$J = \begin{cases} (1+\theta^2)^{-1/2}\left(d_2\theta^{\frac{1+\sqrt{1-4\lambda}}{2}} + d_3\theta^{\frac{1-\sqrt{1-4\lambda}}{2}}\right), & 4\lambda < 1 \\ \theta^{1/2}(1+\theta^2)^{-1/2}\left(d_4 \operatorname{Cos}\sqrt{4\lambda-1}\ \ln\theta + d_5 \operatorname{Sin}\sqrt{4\lambda-1}\ \ln\theta\right), & 4\lambda > 1 \\ \theta^{1/2}(1+\theta^2)^{-1/2}\ (d_6 + d_7\ln\theta) & , 4\lambda = 1 \end{cases}$$

(ii)For

$$\acute{A}/\mu(1+\theta^2) - 1/(1+\theta^2)^2 = \lambda_1\ (\neq 0)$$

we find

$$A = -\lambda_1\mu(\theta+\theta^3/3) - \mu\tan^{-1}\theta + \lambda_2$$

$$J = (1+\theta^2)^{-1/2}\left(\lambda_3 \operatorname{Cos}\sqrt{\lambda_1}\ \theta + \lambda_4 \operatorname{Sin}\sqrt{\lambda_1}\ \theta\right)$$

When $\lambda_1=0$, we find

$A = \mu\tan^{-1}\theta + \lambda_5$

$J=(1+\theta^2)^{-1/2}(\lambda_6+\lambda_7\theta)$

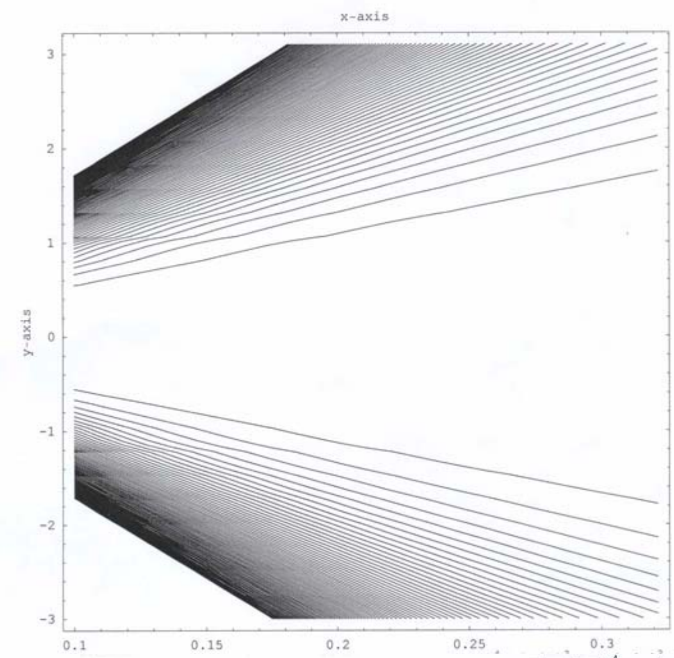

Figure-2: Streamlines pattren for $8+12ArcTan[y/x]+6[\frac{(y/x)^4}{4}+\frac{(y/x)^2}{2}]+6[\frac{(y/x)^2-1}{(y/x)}]=0$

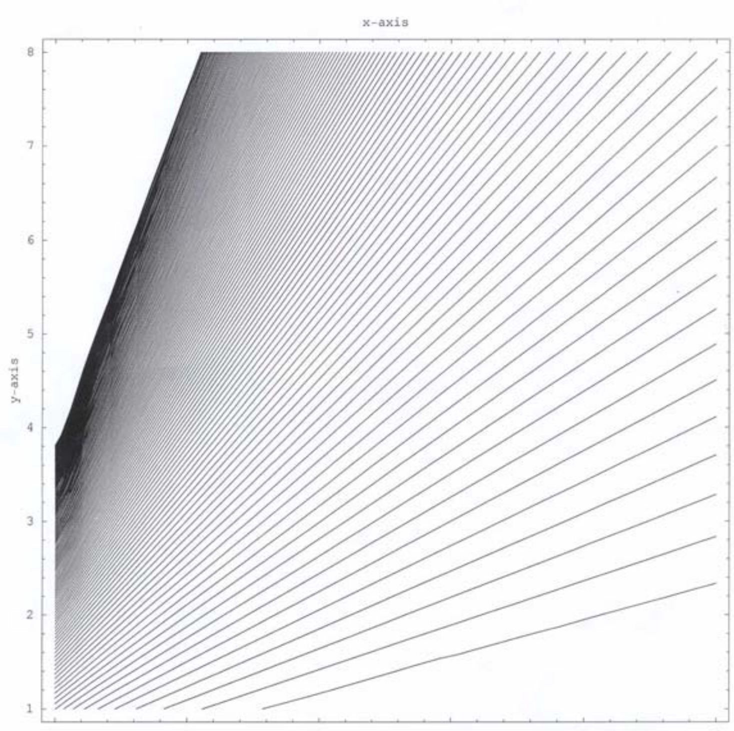

Figure-3:Streamlines pattren for $12+12ArcTan[y/x]+6(\frac{y}{x}+\frac{(y/x)^3}{3})+4(Log[y/x]+\frac{(y/x)^2}{2})-2((y/x)-\frac{x}{y})=0$

In above λ_2, ..., λ_7 are constants

(iii) Solution in terms of Bessel functions

If we assume

$$-\frac{A'}{\mu(1+\theta^2)} - \frac{1}{(1+\theta^2)^2} = \beta^2\gamma^2\theta^{2\gamma-1} - \frac{4p^2\gamma^2-1}{4\theta^2}$$

then Stanley [7]

$$J = \theta^{1/2}(1+\theta^2)^{-1/2} Zp(\beta\theta)$$

$$A = -\mu \tan^{-1}\theta - \mu\beta^2\gamma^2\left[\frac{\theta^{2\gamma}}{2\gamma} + \frac{\theta^{2\gamma+2}}{2\gamma+2}\right] + \frac{\mu(4p^2\gamma^2-1)}{4}\left[\frac{\theta^2-1}{\theta}\right] + D_1$$

wherein β,γ,p,D_1 are constants and Zp is Bessel function of order p*. The streamlines pattern for this solution is given in Figure 2.

iv) If we let

$$-\frac{A'}{\mu(1+\theta^2)} - \frac{1}{(1+\theta^2)^2} = b\,\theta^m$$

then

$$J = \theta^{1/2}(1+\theta^2)^{-1/2} Z_1/(m+2)\left(\frac{2\sqrt{b}\;\theta}{m+2}\right)$$

$$A = -\mu b\left(\frac{\theta^{m+1}}{m+1} + \frac{\theta^{m+3}}{m+3}\right) - \mu \tan^{-1}\theta + D_2$$

wherein b, m, D_2 are constants.

v) Solution in terms of Hermite function

On substituting

$-\acute{A}/\mu(1+\theta^2) - 1/(1+\theta^2)^2 = a_1 - \theta^2$

the equation (25) becomes

$\Phi^{*''} + (a^*_1 - \theta^2)\Phi^* = 0$

whose solution is

$\Phi^* = e^{-\theta^2/2} H_n(\theta)$

wherein $n = (a^*_1 - 1)/2$ and $H_n(\theta)$ is the Hermite –polynomial of degree n. The functions J and A, therefore, are given by

$J = (1+\theta^2)^{-1/2} e^{-\theta^2/2} H_n(\theta)$

$A = -\mu \tan^{-1}\theta + (\mu - a^*_1)\,\theta^3/3 - \mu a^*_1\theta + \mu\theta^5/5 + a^*_2$

wherein $a^*_1 (>0)$, a^*_2 are constants.

(vi) Solution in terms of Whittakers confluent Hypergeometric function.

By putting

$$-\frac{A'}{\mu(1+\theta^2)} - \frac{1}{(1+\theta^2)^2} = -\frac{1}{4} + \frac{K}{\theta} + \frac{1/4 - m^2}{\theta^2}$$

in equation (25), we get

$$\Phi_*'' + \left(-\frac{1}{4} + \frac{K}{\theta} + \frac{1/4 - m^2}{\theta^2}\right)\Phi^* = 0$$

which is Whittaker equation and its solution is

$$\Phi^* = \{ a_3 \theta^{m+1/2} e^{-\theta/2} F(1/2 - K + m;\ 1+2m;\ \theta)$$
$$+ a_4\,\theta^{m-1/2} e^{-\theta/2} F(1/2 - K - m;\ 1-2m;\ \theta)\}$$

wherein K, m, a_3, a_4 are arbitrary constants

The function A is given by

$$A = -\mu \tan^{-1}\theta + \mu/4\,(\theta + \theta^3/3) - K\mu(\ln\theta + \theta^2/2)$$
$$+ (m^2 - 1/4)\mu(\theta - 1/\theta) + a_5$$

wherein a_5 is an arbitrary constant. The streamlines pattern for this solution is depicted in Figure 3.

(vii) If we assume that θ^m is an integral in the complementary function of equation (22), then

$$A = \frac{-\mu m(m-1)}{\theta} + \mu m(m+1)\ \theta + \alpha_1$$

$$J = \alpha_1\theta^m \int \frac{d\theta}{\theta^m(1+\theta^2)} + \alpha_2\theta^m$$

wherein α_1, α_2 are constants. In the expression for J, the indefinite integral can easily be evaluated from the tables of indefinite integrals Bois (1961) for given m.

The equation (22) can be transformed into an integrable form from Sharma and Gupta (1984) by introducing the new independent variable Z through $Z = \tan^{-1}\theta$ and choosing $\acute{A}(Z) = -\mu\alpha_3/Z^2$

the equation (22), on using these, yields

$Z^2 J''(Z) + \alpha_3 J(Z) = 0$

whose solution is

$$
J=\begin{cases}
(\alpha_4+\alpha_5 \ln \tan^{-1}\theta)\,(\tan^{-1}\theta)^{1/2}, & \alpha_3=1/4\\[2ex]
\alpha_5\,(\tan^{-1}\theta)^{\frac{1+\sqrt{1-4\alpha_3}}{2}}+\alpha_7\,(\tan^{-1}\theta)^{\frac{1+\sqrt{1-4\alpha_3}}{2}}, & \alpha_3<1/4\\[2ex]
(\tan^{-1}\theta)^{1/2}\left[\alpha_8 \mathrm{Cos}\,\frac{\sqrt{4\alpha_3-1}}{2}\ln\tan^{-1}\theta+\alpha_9\,\mathrm{Sin}\,\frac{\sqrt{4\alpha_3-1}}{3}\ln\tan^{-1}\theta\right] & \alpha_3>1/4
\end{cases}
$$

wherein α_3, —, α_9 are constants.

The function A is given by

$A = \mu\alpha_3/\tan^{-1}\theta + \alpha_{10}$

wherein α_{10} is an arbitrary constant. In this case the flow pattern is given in Figure 4.

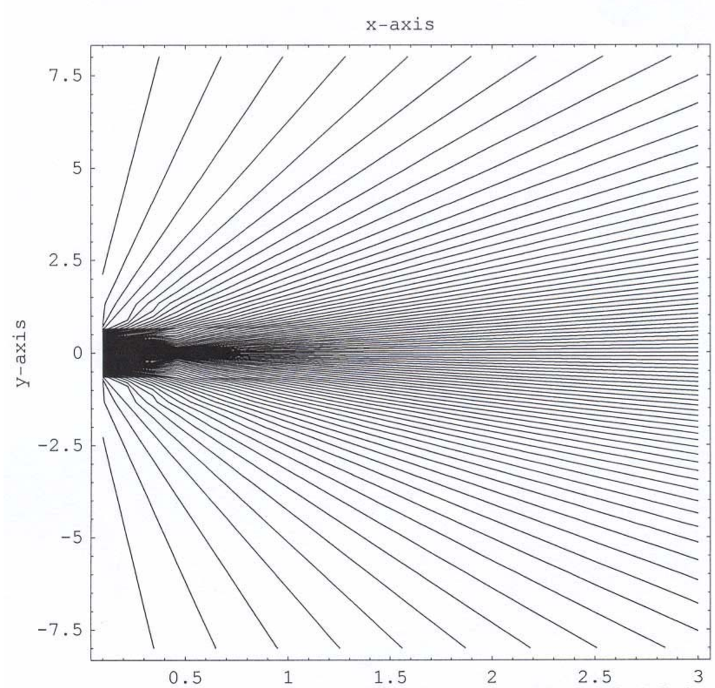

Figure-4:Streamlines pattren for 10-6ArcTan[y/x]=0

4. CONCLUSIONS

Using one parameter group of transformations, the invariants for the Navier-Stroke's equation describing the motion of a compressible fluid of constant viscosity are determined for an arbitrary state equation. These invariants are used to transform the flow equations into a new system of ordinary differential equations whose some exact solutions are determined employing some of the known methods / techniques for the determination of solutions of ordinary differential equations. All the exact solutions involve arbitrary functions or function and this arbitrariness of functions/ function enables us to construct a large number of solutions for the flow equations which are not obtainable through other methods/

techniques. It is indicated that all the exact solutions also hold for the flow of ideal and polytropic gases. For some solutions streamline patterns are also presented.

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